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Supercoherent states for the t–J model

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Abstract. The Cartan structure of the dynamical superalgebra of the t–J model of strongly correlated electrons is found. This is used to define and explicitly construct the coherent states for this algebra and calculate the invariant measure for the completeness relation for the coherent states.

1. Introduction

Supersymmetry (Wess and Bagger 1983, West 1986) has been on the agenda of particle physics for some years now. In condensed matter physics it has made an appearance (Sourlas 1985) from time to time. In this paper we will discuss a potentially very significant appearance, this time of a dynamical superalgebra, in condensed matter physics, and its mathematical description. The recent appearance is in one of the most active areas for many years: the study of metal–insulator transitions (Mott 1974). The interest has been invigorated by the relevance of these transitions to high temperature cuprate superconductors (Anderson 1988). Although much effort has been expended in understanding metal–insulator transitions over the last 30 years there is not much rigorously known concerning these systems. Electron correlation effects are at the heart of the study and Hubbard (1965) made an important contribution by encapsulating the inherent competition between localized and band-like behaviour in his celebrated Hamiltonian, H_{Hubbard} given by

$$H_{\text{Hubbard}} = \sum_{\substack{\langle ij\rangle \\ \sigma}} t c_{i\sigma}^{\dagger} c_{j\sigma} + U \sum_{\sigma, i} n_{i\sigma} n_{i,-\sigma}. \tag{1}$$

Here i and j denote lattice sites which are nearest neighbours, t is a hopping matrix, $c_{i\sigma}^{\dagger}$ creates an electron at site i with spin σ and $n_{i\sigma} = c_{i\sigma}^{\dagger} c_{i\sigma}$. U measures the correlation energy. For $U = 0$ we have band-like behaviour while for $t = 0$ we have the atomic limit. An important characteristic of the cuprate superconducting materials is the presence of antiferromagnetism at half-filling. Consequently, effective Hamiltonians which explicitly have spin–spin interactions are particularly useful. From the Hubbard model (and other more general models which contain degrees of freedom for both oxygen and copper orbitals), the following effective Hamiltonian H_{t-J} can be deduced (see e.g. Pike *et al* 1991):

$$H_{t-J} = P\left(\sum_{\substack{\langle ij\rangle \\ \sigma}} t c_{i\sigma}^{\dagger} c_{j\sigma} + J \sum_{\langle ij\rangle} (\boldsymbol{S}_i \cdot \boldsymbol{S}_j - \tfrac{1}{4}) n_i n_j\right) P. \tag{2}$$

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The Heisenberg spin–spin interaction thus appears explicitly. An important feature is the operator P which is the projector onto states with no double occupation at a site. Both the Hubbard and t–J models (except in 1D) have defied solution. Even in the absence of the J-term the operator $P\sum tc^\dagger_{i\sigma}c_{j\sigma}P$ gives rise in general to a spectrum which is not band-like. The basic operators, in some sense, are no longer $c_{i\sigma}$ but $c_{i\sigma}P$. Such operators do not have the canonical commutation relations of fermionic operators. Since there are no reliable methods for computations for fermions with such constraints, it is important to develop alternative mathematical representations for such operators. We actually meet similar problems when dealing with spin operators in the Heisenberg model. Spin operators, of course, do not have canonical commutation relations either. An interesting approach to spin systems which throws light on the relation between spin and statistics of elementary excitations involves a coherent state representation (Klauder and Skagerstam 1985) of partition functions (Fradkin and Stone 1988).

As a preliminary to the study of the corresponding physics of coherent states for H_{t-J} we will give in this paper a detailed discussion of the mathematics of coherent states appropriate for H_{t-J}. In fact the coherent states are associated with the dynamical algebra of the Hamiltonian. For H_{t-J} it has been noticed only quite recently (Wiegmann 1988) that this algebra is actually a finite dimensional superalgebra (Cornwell 1989). The dynamical superalgebra can in fact be the invariance algebra of H_{t-J} for a certain choice of parameters in H_{t-J} (Sarkar 1990). We will discuss this superalgebra in some detail and construct its highest weight coherent states which will be called supercoherent states. Dynamical superalgebras within the context of strongly correlated electron systems have also been discussed by Montorsi *et al* (1989a, b, c). We will find that it is possible to follow to a large extent the steps used for the purely spin problem.

2. The t–J superalgebra

In order to discuss supersymmetry and superalgebra it is very helpful to introduce Hubbard X-operators. For the t–J model they are defined as follows:

$$X_i^{\alpha\beta} = |\alpha_i\rangle\langle\beta_i| \tag{3}$$

where

$$|\alpha_i\rangle \in \{|0\rangle, |\uparrow\rangle, |\downarrow\rangle\}$$

$|0\rangle$ is the state at site i with no electrons, $|\uparrow\rangle$ is the state with one up spin and $|\downarrow\rangle$ the state with one down spin. In terms of X-operators H_{t-J} can be written without the appearance of the operator P. This is because the X have explicitly incorporated the constraint of no double occupation. In fact

$$H_{t-J} = t\sum_{\substack{\langle ij\rangle\\ \sigma}} X_i^{\sigma 0}X_j^{0\sigma} + \frac{J}{2}\sum_{\substack{\langle ij\rangle\\ \sigma,\sigma'}} X_i^{\sigma\sigma'}X_j^{\sigma'\sigma}\delta_{n_i 1}\delta_{n_j 1} - \frac{J}{2}\sum_{\langle ij\rangle}\delta_{n_i 1}\delta_{n_j 1}. \tag{4}$$

The form of the t-term is self-evident on using the definition of the X-operators. The J-term can be understood from noting that

$$\begin{aligned} J(\boldsymbol{S}_i\cdot\boldsymbol{S}_j - \tfrac{1}{4}) &= \frac{J}{4}(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j - 1) \\ &= \frac{J}{2}(\tfrac{1}{2}(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j + 1) - 1) \end{aligned} \tag{5}$$

(where $\boldsymbol{\sigma}$ denotes the three Pauli matrices). By explicitly considering the effect on the direct product of two spin states it is easy to verify that

$$\tfrac{1}{2}(\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j + 1)$$

permutes the spins at site i and site j. Consequently

$$\tfrac{1}{2}(\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j + 1) = \sum_{\sigma,\sigma'} X_i^{\sigma\sigma'} X_j^{\sigma'\sigma} \tag{6}$$

and we have derived the form of the J-term in (5). The other term in (4) can also be expressed in terms of X-operators (Sarkar 1991). The dynamical algebra is the algebra of the X. Since $X_i^{0\sigma}$ (and $X_i^{\sigma 0}$) change the charge occupation of a site it is natural to regard them as fermionic. The remaining X-operators leave the charge occupation unchanged and so are taken to be bosonic. The algebra (Wiegmann 1988) is then

$$[X_i^{\alpha\beta}, X_j^{\gamma\delta}]_\pm = \delta_{ij}(X_i^{\alpha\delta}\delta_{\beta\gamma} \pm X_i^{\gamma\beta}\delta_{\alpha\delta}). \tag{7}$$

The anticommutator (+) occurs only when both $X_i^{\alpha\beta}$ and $X_j^{\gamma\delta}$ are fermionic. Such algebras which involve both commutators (−) and anticommutators (+) are known as superalgebras. Just as for spin systems it is often convenient to represent the superalgebra operators in terms of bilinears of harmonic oscillators (Bars and Günaydin 1983). However, although this is frequently used the constraint of no double occupancy has to be enforced at each site explicitly and in a calculation of the partition function this necessitates the introduction of Lagrange multipliers, one for each site (Bickers 1987). One of the motivations of the method of coherent states is to incorporate the constraint in a more intrinsic manner. None the less we will use harmonic oscillator representations of the algebra in order to help unravel its Cartan structure (Georgi 1982).

For clarity we will consider a harmonic oscillator representation which is more general than the t–J algebra. For a system of n bosonic harmonic oscillators b_i $(i = 1, \ldots, n)$ and m fermionic harmonic oscillators f_γ $(\gamma = 1, \ldots, m)$ a complete set A of bilinears is

$$A = \{S_{\alpha i}^{(+)}, S_{i\alpha}^{(-)}, Z_{ij}, Y_{\alpha\beta}\} \tag{8}$$

with

$$S_{\alpha i}^{(+)} = f_\alpha b_i^\dagger = (S_{i\alpha}^{(-)})^\dagger \tag{9a}$$

$$Z_{ij} = b_i b_j^\dagger \tag{9b}$$

and

$$Y_{\alpha\beta} = f_\alpha f_\beta^\dagger . \tag{9c}$$

The S-bilinears (being odd in fermionic oscillators) are of fermionic type while the Z- and Y-operators are equally clearly bosonic. From the representation theory of ordinary Lie algebras it is known that $\{Z_{ij} \mid i, j = 1, \ldots, n\}$ generates U(n) and $\{Y_{\alpha\beta} \mid \alpha, \beta = 1, \ldots, m\}$ generates U(m). It is customary to denote A by U(n/m). These algebras may serve as large n generalizations of the t–J algebra. We will now write the Hubbard operators for the t–J model in terms of the operators of U(1/2).

$$\begin{aligned} &X^{0\uparrow} = S_{11}^{(+)} \qquad X^{0\downarrow} = S_{21}^{(+)} \\ &X^{00} = Z - 1 \\ &X^{\uparrow\uparrow} = 1 - Y_{11} \qquad X^{\downarrow\downarrow} = 1 - Y_{22} \end{aligned} \tag{10}$$

and

$$X^{\uparrow\downarrow}=1-Y_{21}.$$

Analogous relations for the corresponding Hermitian conjugate operators hold. The condition for no double occupancy is the completeness relation

$$X^{00}+X^{\uparrow\uparrow}+X^{\downarrow\downarrow}=1 \tag{11}$$

and this translates for the harmonic oscillator representation into a restriction on the Hilbert space (Bars 1985)

$$f_1^\dagger f_1+f_2^\dagger f_2+b^\dagger b=1. \tag{12}$$

(This equation should not be regarded as an operator identity.)

We can calculate the commutator or anticommutator of the X-operators using (9), (10) and (12) and it is easy to verify that we obtain the algebra of (7).

It is possible to consider different harmonic oscillator representations of the t–J algebra but the details of the representation are not of primary interest. We only use this representation because it serves to guide us in rewriting the t–J algebra as close as possible to a Cartan form. The Cartan form is necessary for the construction of simple coherent states. When a standard Lie algebra G is written in Cartan form the generators are divided into two sets. One is a maximally commuting set $\{H_i\}$ and the other a set of 'lowering' or 'raising' operators $\{E\boldsymbol{\alpha}\}$ where $\boldsymbol{\alpha}$ is a real 'root' vector (Georgi 1982). The defining commutators for a Lie algebra in the standard Cartan form are

$$[H_i, H_j]=0 \tag{13a}$$

$$[H_i, E_\alpha]=\alpha_i E_\alpha \tag{13b}$$

$$[E_{\boldsymbol{\alpha}}, E_{-\boldsymbol{\alpha}}]=\alpha_i H_i \tag{13c}$$

and

$$\begin{aligned}[E_{\boldsymbol{\alpha}}, E_{\boldsymbol{\beta}}] &= N_{\boldsymbol{\alpha},\boldsymbol{\beta}} E_{\boldsymbol{\alpha}+\boldsymbol{\beta}} && \text{if } \boldsymbol{\alpha}+\boldsymbol{\beta}\neq 0 \text{ and } \boldsymbol{\alpha}+\boldsymbol{\beta} \text{ is a root}\\ &=0 && \text{otherwise.}\end{aligned} \tag{13d}$$

A state $|\rangle$ is said to have a (vector) weight λ if

$$H_i|\rangle=\lambda_i|\rangle. \tag{14}$$

In the vector space containing roots, a root $\boldsymbol{\alpha}$ is labelled positive if it is above (some fixed) hyperplane which does not contain any root. A state $|h\rangle$ such that

$$E_{\boldsymbol{\alpha}}|h\rangle=0 \tag{15}$$

for all positive roots $\boldsymbol{\alpha}$, is said to have highest weight. We now have all the supplementary concepts necessary to define coherent states for usual Lie groups. A coherent state $|g\rangle$ is basically defined by

$$|g\rangle=g|h\rangle \tag{16}$$

where $g\in G^{\mathrm{c}}$, the complexification of the Lie group.

A highest weight state does not strictly have to be used. However, its use leads to simplifications because a group element can be decomposed into a product of three terms consisting of (i) an exponential of a linear combination of lowering operators,

(ii) an exponential of a linear combination of operators in the Cartan subalgebra and (iii) an exponential of a linear combination of raising operators (Perelomov 1986). When operating on a highest weight state the last of these exponentials is equivalent to a unit operator, and is the rationale for using highest weight states. In quantum mechanics, states which differ by a phase are physically equivalent. Consequently, an equivalence class of coherent states may be considered, elements in the class differing merely by a phase. The g occurring in (16) should then be regarded as an element of G/H where H is the isotropy group of $|h\rangle$.

In the representation of the partition function using Trotter's formula (Klauder and Skagerstam 1985), the resolution of the identity is the crucial relation that we need. In terms of $D\mu(g)$, the group invariant measure, it reads

$$\int D\mu(g)|g\rangle\langle g| = 1. \tag{17}$$

If we were to parametrize the coherent states with G^c/H then the appropriate measure would be induced on G^c/H by $D\mu(g)$. It is important to note that $|g\rangle\langle g|$ is independent of elements of H, and in the superalgebra case it will actually be more convenient to use (17) with $g \in G^c$.

The remainder of the paper will concentrate on the appropriate generalization of $D\mu(g)$ necessary for the t–J algebra. An explicit construction of the measure will be given.

3. The *t–J* supergroup

Much of the understanding of the structure of superalgebras and supergroups has been pioneered by Berezin (1987) and Kac (1977). The geometric aspect, which is our primary interest, is often presented in a complicated and formal way which makes it harder to use in an application to physics. Hence we will give a fairly self-contained discussion which derives relations necessary for the t–J algebra without relying on general theorems.

In order to mimic for the t–J algebra the Cartan structure, familiar from the theory of Lie algebras, we will rewrite it in the form

$$[\hat{e}_i, \hat{f}_j] = \delta_{ij}\hat{h}_i + (1-\delta_{ij})(u_{ij}\hat{e} + v_{ij}\hat{f}) \tag{18a}$$

$$[\hat{h}_i, \hat{h}_j] = 0 \tag{18b}$$

$$[\hat{h}_i, \hat{e}_j] = a_{ij}\hat{e}_j \tag{18c}$$

and

$$[\hat{h}_i, \hat{f}_j] = -a_{ij}\hat{f}_j. \tag{18d}$$

$[a, b]$ denotes $[a, b]_+$ if both a and b are fermionic operators and $[a, b]_-$ otherwise. In (18) $\hat{h}_i$ is bosonic. $\hat{e}_i$ and $\hat{f}_i$ are either both fermionic or both bosonic. The set of integers i for fermionic $\hat{e}_i$ and $\hat{f}_i$ will be denoted by F. (The indices i and j here should not be confused with lattice points. We have used lower-case notation for the elements of the superalgebra in order to distinguish this discussion from that for Lie algebras.) Clearly the role of the Cartan subalgebra is played by $\{\hat{h}_i\}$. After a little experimentation we find the correspondence

$$\begin{aligned} &\hat{e}_i \leftrightarrow S_\alpha^{(+)} && i \in F \\ &\hat{f}_i \leftrightarrow S_\alpha^{(-)} && i \in F \\ &\hat{h}_i \leftrightarrow b^+ b - f_\alpha^+ f_\alpha \\ &\hat{h} \leftrightarrow b^+ b \\ &\hat{e} \leftrightarrow f_1 f_2^+ \\ &\hat{f} \leftrightarrow f_2 f_1^+ . \end{aligned} \tag{19}$$

(The $\hat{f}$ operators are of course quite distinct from the f operators and in the expression for $\hat{h}_i$ there is no summation convention for α being used.) As an example of the calculation of a structure constant of the algebra we note that

$$\begin{aligned} [\hat{h}_i, \hat{e}_j] &\leftrightarrow [h_\alpha, S_\beta^{(+)}] \\ &= (1+\delta_{\alpha\beta}) S_\beta^{(+)} \end{aligned} \tag{20}$$

and so

$$a_{ij} \leftrightarrow 1 + \delta_{\alpha\beta}. \tag{21}$$

If we base the highest weight state on the ladder operators $\hat{e}_i$ and $\hat{e}$ in the Hilbert space constrained by (12), then it is easy to verify that the highest weight state is $|0\rangle$. On the other hand if the root space hypersurface is chosen such that $\hat{f}_i$ and $\hat{f}$ are the raising operators, then the highest weight state is $|\uparrow\rangle$.

In order to construct coherent states we need to construct a group element corresponding to a superalgebra. We recall that standard Lie group elements connected to the identity are obtained by exponentiating Lie algebra elements. There is thus a one-to-one correspondence between such elements of the group and points in R^l where l is the order of the group. An analogous procedure for supergroups requires a Grassmann generalization of R^l (Cornwell 1989). A relevant generalization of R is the algebra RB_M where M is the number of Grassmann generators ζ_j. RB_M consists of elements which are sums of products of Grassmann generators with coefficients in R multiplying the products. If a term does not involve any Grassmann elements it reduces to an element of R. The subalgebra consisting of elements which are sums of even products of Grassmann generators is denoted by RB_{M0} and the remaining elements by RB_{M1}. A natural generalization of R^l is $RB_M^{l,n}$ which is defined as $(RB_{M0})^l \otimes (RB_{M1})^n$. In a similar way the complex algebra $CB_M^{l,n}$ may be defined. Instead of defining a supergroup in terms of the super Lie algebra we will do the inverse procedure. First we need the concept of a supermatrix (Cornwell 1989). An even $(p/q)\times(p/q)$ supermatrix M has the form

$$M = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \tag{22}$$

with P a $p\times p$ matrix with elements in CB_{M0}, Q a $p\times q$ matrix with elements in CB_{M1}, R a $q\times p$ matrix with elements in CB_{M1} and S a $q\times q$ matrix with elements in CB_{M0}. Such a set of supermatrices forms a group with respect to matrix multiplication. In order to introduce the associated super-Lie algebra we need to have a norm $\|\ \|$ on supermatrices. A suitable norm is given by

$$\|M\| = \sum_{j=1}^{p+q} \sum_{k=1}^{p+q} \|M_{jk}\| \tag{23}$$

and this induces a distance function d in the space of matrices through

$$d(M, M') = \|M - M'\|. \tag{24}$$

(In (23) $\|M_{jk}\|$ is evaluated in terms of a suitable norm in CB_M.)

Consequently M is in a neighbourhood of the identity of 1_{p+q} if $d(M, 1_{p+q})$ is sufficiently small. M can then be parametrized by an element $(x^1, x^2, \ldots, x^m, \psi^1, \ldots, \psi^n)$ of $RB_M^{m,n}$. Such elements form a Lie supergroup which is said to have even dimension m and odd dimension n. 1_{p+q} is labelled by the 0 element of $RB_M^{m,n}$. The inverse procedure to that of exponentiating a Lie algebra element to obtain a Lie group element gives the following generators:

$$L^j_\mu = \frac{\partial M(x, \psi)}{\partial x^j_\mu}\bigg|_{x=0,\,\psi=0} \tag{25}$$

and

$$K^k_\mu = \frac{\partial M(x, \psi)}{\partial \psi^k_\mu}\bigg|_{x=0,\,\psi=0}. \tag{26}$$

Here x^j_μ and ψ^j_μ are real numbers related to x and ψ through

$$x^j = \sum_{\substack{\{\mu\}\\ \text{even}}} x^j_\mu \zeta_\mu \tag{27}$$

and

$$\psi^j = \sum_{\substack{\{\mu\}\\ \text{odd}}} \psi^j_\mu \zeta_\mu \tag{28}$$

where μ indexes a monotonically ordered set of distinct integers in the set $\{1, \ldots, M\}$, and ζ_μ is a product $\Pi\zeta_i$ where i is taken from a subset of the set indexed by μ. If μ is, for example, $\{1, 2, M\}$ then

$$\zeta_\mu = \zeta_1\zeta_2\zeta_M \tag{29}$$

and μ is odd.

In section 2 we showed that the t–J algebra is $U(1|2:C)$. The corresponding supergroup element is known (Cornwell 1989) to satisfy

$$M^+M = 1_3 \tag{30}$$

where

$$M^+ = \begin{pmatrix} \tilde{P}^\# & \tilde{R}^\# \\ \tilde{Q}^\# & \tilde{S}^\# \end{pmatrix} \tag{31}$$

P is a 1×1 matrix, R is a 1×2 matrix, Q is a 2×1 matrix and S is a 2×2 matrix. The symbol '$\sim$' indicates matrix transpose and $\#$ is a Grassmann adjoint operation

$$\begin{aligned} \zeta^\#_\mu &= \zeta_\mu && \text{if } \zeta_\mu \text{ is even} \\ &= -\mathrm{i}\zeta_\mu && \text{if } \zeta_\mu \text{ is odd.} \end{aligned} \tag{32}$$

The $\#$ operation also requires the taking of the complex conjugate of all complex numbers.

We will proceed with the construction of coherent states. For the t–J model a basis for the matrix representation is $|0\rangle$, $|\uparrow\rangle$ and $|\downarrow\rangle$, the basis element $|0\rangle$ having bosonic grading and $|\uparrow\rangle$ and $|\downarrow\rangle$ fermionic grading. The elements of the isotropy group H of $|0\rangle$ by inspection have the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} U & W \\ 0 & V \end{pmatrix}. \tag{33}$$

Similarly for $|\uparrow\rangle$, a typical isotropy group element is

$$\begin{pmatrix} a_{11} & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & 0 & a_{33} \end{pmatrix}.$$

Let us identify the superalgebra generators $\{L_j\}$ and $\{K_\mu\}$ which are bosonic and fermionic respectively. We have

$$\{L_j\} = \{b^\dagger b, f_1 f_2^\dagger, f_2 f_1^\dagger, f_1^\dagger f_1, f_2^\dagger f_2\} \tag{34}$$

and

$$\{K_\mu\} = \{f_\alpha b^\dagger, \ bf_\alpha^\dagger, \alpha = 1, 2\}. \tag{35}$$

A general supergroup element g, from our discussion, has the form

$$g = \exp(\psi_\mu K_\mu + x_i L_i) \tag{36}$$

where $\psi_\mu \in CB_{M1}$ and $x_i \in CB_{M0}$. From (30) we deduce that the exponent satisfies

$$(\psi_\mu K_\mu + x_i L_i)^{\#} + (\psi_\mu K_\mu + x_i L_i) = 0. \tag{37}$$

Equation (37) for superalgebras is the analogue of anti-Hermiticity for Lie algebras and ensures unitarity.

It will be convenient to reparametrize g in (38). We will write x_i as $y_i' + y_i$ where y_i' is nilpotent but y_i is an ordinary complex number. Now from the group properties we know that

$$\exp(\psi_\mu K_\mu + x_i L_i)\exp(-y_j L_j) = \exp(\psi_\mu' K_\mu + x_i' L_i) \tag{38}$$

for some ψ_μ' and x_i'. The explicit form of ψ_μ' and x_i' can be found from the Campbell–Hausdorff formula which we will require frequently in our later calculations. We will produce the formula where we first use it. At this stage we just want to point out that x_i' is nilpotent. Indeed

$$\begin{aligned} \exp(\psi_\mu K_\mu + x_i L_i) &= \exp(\psi_\mu K_\mu + y_i L_i + y_i' L_i) \\ &= \sum_{j=0}^{\infty} \frac{1}{j!}(\psi_\mu K_\mu + y_i L_i + y_i' L_i)^j \\ &= \sum_{j=0}^{\infty} \frac{1}{j!}(y_i L_i)^j + R \\ &= \exp(y_i L_i) + R. \end{aligned} \tag{39}$$

Since R can only be a polynomial function of ψ_μ and y_i' with no constant term it is nilpotent.

From (39) we deduce that

$$\exp(\psi_\mu K_\mu + x_i L_i)\exp(-y_j L_j) = 1 + R\exp(-y_j L_j). \tag{40}$$

Since R is nilpotent, $R\exp(-y_j L_j)$ is also nilpotent. For compactness of notation let

$$\Psi \equiv \psi_\mu' K_\mu + x_i' L_i.$$

Now

$$\exp\Psi = 1 + f(\Psi) \tag{41}$$

where

$$f(\Psi)=\Psi+\frac{\Psi^2}{2!}+\ldots+\frac{\Psi^n}{n!}+\ldots.$$

For $f(\Psi)$ to be nilpotent it is necessary and sufficient that Ψ (i.e. $\psi'_\mu K_\mu+x'_i L_i$) is nilpotent.

Consequently x'_i also has to be nilpotent. However, for the construction of coherent states it is useful to make a further simplification. As representative members of elements in G/H we will take

$$\exp(\psi_\mu K_\mu)\exp(x_i L_i)$$

where x_i are complex numbers. This is permissible as we will see from applying the Campbell-Hausdorff (CH) formula in evaluating products of exponentials. In fact the CH formula is

$$\exp(u)\exp(v)=\exp(v+\tfrac{1}{2}R_v(1+\coth\tfrac{1}{2}R_v)u) \tag{42}$$

where R_v is an operator such that

$$R_v u=[u,v]. \tag{43}$$

It is easy to check that

$$\tfrac{1}{2}R_v(1+\coth\tfrac{1}{2}R_v)u=u+\tfrac{1}{2}[u,v]+\tfrac{1}{12}[[u,v],v]+\ldots. \tag{44}$$

Let us first consider coherent states based on $|0\rangle$. It is convenient to parametrize elements h of H as

$$h=\exp(x_j L_j)\exp(aX^{00}). \tag{45}$$

Here we have chosen the notation that L_j are generators of SU(2); x_j and a are even elements of the Grassmann algebra. Now gh where

$$g=\exp(\psi_\mu K_\mu+y'_i L_i)\exp(y_j L_j) \tag{46}$$

(with y' nilpotent and y_j complex) can be written as

$$gh=\exp(\psi_\mu K_\mu+y'_i L_i)\exp(Z_k L_k+aX^{00})\exp(y_j L_j) \tag{47}$$

where

$$\exp(Z_k L_k)=\exp(y_j L_j)\exp(x_i L_i)\exp(-y_k L_k). \tag{48}$$

By using the freedom in choosing Z_k and a, the CH formula implies that

$$\exp(\psi_\mu K_\mu+y'_i L_i)\exp(Z_k L_k+aX^{00})=\exp(\psi'_\mu K_\mu) \tag{49}$$

for some ψ'_μ.

Hence a representative gh in a coset has the form

$$\exp(\psi'_\mu K_\mu)\exp(y_j L_j).$$

A similar reasoning can be made for coherent states based on $|\uparrow\rangle$.

4. The invariant measure

In the construction of $D\mu(g)$ for U(1/2; C) we will follow a similar procedure to that used in the theory of ordinary Lie groups. For compactness of notation (and resultant clarity) we give the main steps in our discussion for a general finite dimensional supergroup with m fermionic generators $\{K_\mu\}$ and n bosonic generators $\{L_i\}$. $D\mu(g)$ can be expressed in terms of right-translation vectors. The basic right translations are

$$\exp(\zeta_\mu K_\mu)\exp(x_j L_j) y_k L_k \equiv y \cdot d^{(1)} \exp(\zeta \cdot K)\exp(x \cdot L) \tag{50}$$

and

$$\exp(\zeta_\mu K_\mu)\exp(x_i L_i)\alpha_\nu K_\nu \equiv \alpha \cdot d^{(2)} \exp(\zeta_\mu K_\mu)\exp(x_i L_i) \tag{51}$$

$d^{(1)}$ and $d^{(2)}$ are vector fields and, in component form, may be written as

$$d_j^{(1)} = d_{j\nu}^{(1)} \frac{\partial}{\partial \zeta_\nu} + \bar{d}_{jk}^{(1)} \frac{\partial}{\partial x_k} \tag{52}$$

and

$$d_\mu^{(2)} = d_{\mu\nu}^{(2)} \frac{\partial}{\partial \zeta_\nu} + \bar{d}_{\mu j}^{(2)} \frac{\partial}{\partial x_j}. \tag{53}$$

(From now on we will assume a summation convention on repeated indices.) In terms of the $d^{(1)}$ and $d^{(2)}$ matrices, the invariant measure is given by the superdeterminant (Berezin 1987).

$$D\mu = \begin{vmatrix} d^{(2)} & \bar{d}^{(2)} \\ d^{(1)} & \bar{d}^{(1)} \end{vmatrix} \mathrm{d}\zeta_1 \ldots \mathrm{d}\zeta_n \, \mathrm{d}x_1 \ldots \mathrm{d}x_m. \tag{54}$$

We will now calculate $d^{(1)}$, $\bar{d}^{(1)}$, $d^{(2)}$ and $\bar{d}^{(2)}$.

The definitions of $\partial/\partial\zeta_\nu$ and $\partial/\partial x_j$ are the usual ones, i.e.

$$\begin{aligned} \delta\zeta_\nu \frac{\partial}{\partial \zeta_\nu}(\mathrm{e}^{\zeta\cdot K}\,\mathrm{e}^{x\cdot L}) &\simeq \mathrm{e}^{(\zeta+\delta\zeta)\cdot K}\,\mathrm{e}^{x\cdot L} - \mathrm{e}^{\zeta\cdot K}\,\mathrm{e}^{x\cdot L} \\ &= \mathrm{e}^{\zeta\cdot K}(\mathrm{e}^{-\zeta\cdot K}\,\mathrm{e}^{(\zeta+\delta\zeta)\cdot K} - 1)\,\mathrm{e}^{x\cdot L} \end{aligned} \tag{55}$$

and

$$\begin{aligned} \delta x_j \frac{\partial}{\partial x_j}(\mathrm{e}^{\zeta\cdot K}\,\mathrm{e}^{x\cdot L}) &\simeq \mathrm{e}^{\zeta\cdot K}\,\mathrm{e}^{(x+\delta x)\cdot L} - \mathrm{e}^{\zeta\cdot K}\,\mathrm{e}^{x\cdot L} \\ &= \mathrm{e}^{\zeta\cdot K}\,\mathrm{e}^{x\cdot L}(\mathrm{e}^{-x\cdot L}\,\mathrm{e}^{(x+\delta x)\cdot L} - 1). \end{aligned} \tag{56}$$

We can use the CH formula to evaluate the relevant products of these exponentials. An important simplification occurs since products of Grassmann variables of order $(M+1)$ or higher vanish. It is easy to show that

$$\delta\zeta_\nu \partial_\nu(\mathrm{e}^{\zeta\cdot K}\,\mathrm{e}^{x\cdot L}) = \delta\zeta_\nu\,\mathrm{e}^{\zeta\cdot K}(b_{\nu\mu}K_\mu + a_{\nu j}L_j)\,\mathrm{e}^{x\cdot L} \tag{57}$$

where $a_{\nu j}$ and $(b_{\nu\mu} - \delta_{\nu\mu})$ are both nilpotent. On using

$$\mathrm{e}^{-x\cdot L}K_\mu\,\mathrm{e}^{x\cdot L} = h_{\mu\nu}K_\nu \tag{58}$$

and

$$\mathrm{e}^{-x\cdot L}L_j\,\mathrm{e}^{x\cdot L} = r_{jk}L_k \tag{59}$$

(for suitable $h_{\mu\nu}$ and r_{jk}) to be calculated later (57) can be put into a form

$$\delta\zeta^\nu \partial_\nu(\mathrm{e}^{\zeta\cdot K}\,\mathrm{e}^{x\cdot L}) = \delta\zeta^\nu\,\mathrm{e}^{\zeta\cdot K}\,\mathrm{e}^{x\cdot L}(b_{\nu\mu}h_{\mu\lambda}K_\lambda + a_{\nu j}r_{jk}L_k). \tag{60}$$

Similar type of arguments can show that

$$\delta x_j \partial_j (e^{\zeta\cdot K}\, e^{x\cdot L}) = \delta x_j\, e^{\zeta\cdot K}\, e^{x\cdot L}\, e_{jk} L_k \tag{61}$$

where e_{jk} is a function of x. Now (50) translates into

$$y\cdot d^{(1)}\, e^{\zeta\cdot K}\, e^{x\cdot L}$$

$$= y_j d^{(1)}_{j\nu}\, e^{\zeta\cdot K}\, e^{x\cdot L}(b_{\nu\mu} h_{\mu\lambda} K_\lambda + a_{\nu i} r_{ik} L_k) + y_j \bar{d}^{(1)}_{jk}\, e^{\zeta\cdot K}\, e^{x\cdot L} e_{ki} L_i. \tag{62}$$

Since (50) and (62) both have to be true it is necessary that

$$d^{(1)}_{j\nu} = 0 \tag{63}$$

(i.e. the coefficient of K_ν has to vanish). Hence we have

$$\bar{d}^{(1)}_{jk} e_{ki} = \delta_{ji}. \tag{64}$$

Similarly

$$e^{\zeta\cdot K}\, e^{x\cdot L} \alpha\cdot K = \alpha_\lambda d^{(2)}_{\lambda\nu}\, e^{\zeta\cdot K}\, e^{x\cdot L}(b_{\nu\mu} h_{\mu\nu'} K_{\nu'} + a_{\nu j} r_{jk} L_k) + \alpha_\lambda d^{(2)}_{\lambda j}\, e^{\zeta\cdot K}\, e^{x\cdot L} e_{jk} L_k \tag{65}$$

and so

$$d^{(2)}_{\mu'\nu} a_{\nu j} r_{jk} + d^{(2)}_{\mu' j} e_{jk} = 0 \tag{66}$$

and

$$d^{(2)}_{\mu'\nu} b_{\nu\mu} h_{\mu\nu'} = \delta_{\mu'\nu'}. \tag{67}$$

Equations (64), (66) and (67) permit the calculation of all non-zero $d^{(1)}$ and $d^{(2)}$. The auxiliary quantities in terms of which they are expressed (such as $b_{\nu\mu}$) will be calulated in the appendix.

A superdeterminant Δ of a $p/q \times p/q$ supermatrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is defined to be (Cornwell 1989)

$$\Delta = (\det(A - BD^{-1}C))(\det D^{-1}). \tag{68}$$

Since $d^{(1)}$ has vanished we have

$$\begin{vmatrix} d^{(2)} & d^{(1)} \\ \bar{d}^{(2)} & \bar{d}^{(1)} \end{vmatrix} = \frac{\det d^{(2)}}{\det \bar{d}^{(1)}} \tag{69}$$

(det denotes a standard determinant).

From (64) and (67) it follows that

$$\frac{1}{\det \bar{d}^{(1)}} = \det e \tag{70}$$

and

$$\det d^{(2)} = (\det b)^{-1} (\det h)^{-1}.$$

$\det e$ is just the usual measure corresponding to the Lie group associated with $\{L_j\}$ and so is already familiar. In the appendix we obtain simple expressions for $\det b$ and $\det h$. From (54) we now have the invariant measure, and hence the resolution of the identity.

Although we have examined the t–J algebra in detail it is rather typical of the other algebras in the study of strongly correlated systems. So far the bulk of such studies have relied on the harmonic oscillator representations which have constraints that have to be imposed and so are not intrinsic. An impediment to the application of coherent state methods has been the lack of a discussion of explicit mathematical properties of coherent states for the comparatively unfamiliar t–J type algebras. Our note, we hope, will fill this gap.

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Appendix

It is convenient to work with the following form of the bosonic operators:

$$L_1=\frac{\mathrm{i}}{2}(f_1^\dagger f_2+f_2^\dagger f_1) \tag{A1}$$

$$L_2=\tfrac{1}{2}(f_1^\dagger f_2-f_2^\dagger f_1) \tag{A2}$$

$$L_3=\frac{\mathrm{i}}{2}(f_1^\dagger f_1-f_2^\dagger f_2) \tag{A3}$$

$$L_4=\mathrm{i}b^\dagger b \tag{A4}$$

and

$$L_5=\mathrm{i}. \tag{A5}$$

$-\mathrm{i}L_1$, $-\mathrm{i}L_2$ and $-\mathrm{i}L_3$ are the usual SU(2) generators and satisfy the algebra

$$[L_j, L_k]=\varepsilon_{jkm}L_m. \tag{A6}$$

Moreover we will choose

$$K_1=b^\dagger f_1 \tag{A7}$$

$$K_2=b^\dagger f_2 \tag{A8}$$

$$K_3=f_1^\dagger b \tag{A9}$$

and

$$K_4=f_2^\dagger b. \tag{A10}$$

We will first calculate $h_{\mu\nu}$ (defined by (56)).

$$\begin{aligned}\mathrm{e}^{-x\cdot L}K_\mu\,\mathrm{e}^{x\cdot L} &= K_\mu+[-x\cdot L, K_\mu]+\frac{1}{2!}[-x\cdot L,[-x\cdot L, K_\mu]] \\ &\quad+\ldots+\frac{1}{n!}[-x\cdot L,\ldots[-x\cdot L,[-x\cdot L, K_\mu]]\ldots]+\ldots\end{aligned} \tag{A11}$$

If we write

$$[L_i, K_\mu] = M^i_{\mu\nu} K_\nu \tag{A12}$$

then from (69)

$$e^{-x\cdot L} K_\mu \, e^{x\cdot L} = (e^{-N})_{\mu\nu} K_\nu \tag{A13}$$

where

$$N_{\mu\nu} = x_i M^i_{\mu\nu} \tag{A14}$$

$$\therefore \qquad h_{\mu\nu} = (e^{-N})_{\mu\nu}. \tag{A15}$$

The structure function $M^i_{\mu\nu}$ is determined by (7).

For $b_{\mu\nu}$ we have to calculate

$$e^{-\zeta\cdot K} \, e^{(\zeta+\delta\zeta)\cdot K}$$

and from the CH formula we can deduce that

$$\begin{aligned} e^{-\zeta\cdot K} \, e^{(\zeta+\delta\zeta)\cdot K} &= \exp(\delta\zeta\cdot K + \tfrac{1}{2}[-\zeta\cdot K, \delta\zeta\cdot K] - \tfrac{1}{12}[[\zeta\cdot K, \delta\zeta\cdot K], \zeta\cdot K] \\ &\quad + \tfrac{1}{720}[[[[\zeta\cdot K, \delta\zeta\cdot K], \zeta\cdot K], \zeta\cdot K], \zeta\cdot K]). \end{aligned} \tag{A16}$$

(The exponent has terminated at a fourth-order term since the number of fermionic generators is 4.) On identifying the commutators which lead to fermionic terms we find

$$\begin{aligned} \delta\zeta^\nu b_{\nu\mu} K_\mu = \tfrac{1}{720}&[[[[\zeta\cdot K, \delta\zeta\cdot K], \zeta\cdot K], \zeta\cdot K], \zeta\cdot K] \\ &- \tfrac{1}{12}[[\zeta\cdot K, \delta\zeta\cdot K], \zeta\cdot K] + \delta\zeta\cdot K. \end{aligned} \tag{A17}$$

Further detailed calculations give

$$b_{11} = -\tfrac{1}{12}(\psi_2^\dagger\psi_2 + \tfrac{1}{12}\psi_1^\dagger\psi_2^\dagger\psi_1\psi_2) + 1 \tag{A18a}$$

$$b_{12} = \tfrac{1}{12}\psi_2^\dagger\psi_1 \tag{A18b}$$

$$b_{13} = 0 \tag{A18c}$$

$$b_{14} = \tfrac{1}{6}\psi_1\psi_2 \tag{A18d}$$

$$b_{21} = \tfrac{1}{12}\psi_1^\dagger\psi_2 \tag{A18e}$$

$$b_{22} = -\tfrac{1}{12}(\psi_1^\dagger\psi_1 + \tfrac{1}{12}\psi_1^\dagger\psi_2^\dagger\psi_1\psi_2) + 1 \tag{A18f}$$

$$b_{23} = \tfrac{1}{6}\psi_2\psi_1 \tag{A18g}$$

$$b_{24} = 0 \tag{A18h}$$

$$b_{31} = 0 \tag{A18i}$$

$$b_{32} = -\tfrac{1}{6}\psi_1^\dagger\psi_2^\dagger$$

$$b_{33} = -\tfrac{1}{12}(\psi_2^\dagger\psi_2 + \tfrac{1}{12}\psi_1^\dagger\psi_2^\dagger\psi_1\psi_2) + 1 \tag{A18j}$$

$$b_{34} = \tfrac{1}{12}\psi_1^\dagger\psi_2 \tag{A18k}$$

$$b_{41} = -\tfrac{1}{6}\psi_2^\dagger\psi_1^\dagger \tag{A18l}$$

$$b_{42} = 0 \tag{A18m}$$

$$b_{43} = \tfrac{1}{12}\psi_2^\dagger\psi_1 \tag{A18n}$$

$$b_{44} = -\tfrac{1}{12}(\psi_1^\dagger\psi_1 + \tfrac{1}{12}\psi_1^\dagger\psi_2^\dagger\psi_1\psi_2) + 1 \tag{A18o}$$

where we have used the notation

$$\zeta_1 = \psi_1^\dagger \qquad \zeta_2 = \psi_2^\dagger \qquad \zeta_3 = \psi_1 \qquad \text{and} \qquad \zeta_4 = \psi_2. \tag{A19}$$

Using the relation

$$\det = \exp \operatorname{tr} \log \tag{A20}$$

we have

$$\begin{aligned} \det h &= \exp \operatorname{tr} \log \mathrm{e}^{-N} \\ &= \exp(-\operatorname{tr} N) \\ &= \exp(-x_i M^i_{\mu\mu}). \end{aligned} \tag{A21}$$

A straightforward calculation (on using the explicit expressions in (A18)) gives

$$\begin{aligned} \det b &= 1 - \tfrac{1}{6}(\psi_1^\dagger\psi_1 + \psi_2^\dagger\psi_2) - \tfrac{1}{8}\psi_1^\dagger\psi_2^\dagger\psi_1\psi_2 \\ &= \exp(-\tfrac{1}{6}(\psi_1^\dagger\psi_1 + \psi_2^\dagger\psi_2) - \tfrac{7}{72}\psi_1^\dagger\psi_2^\dagger\psi_1\psi_2) \end{aligned} \tag{A22}$$

$\det e$ gives the measure corresponding to the bosonic Lie subgroup. e_{jk} is calculated from (61) or equivalently by finding δa in

$$\mathrm{e}^{\delta a} = \exp(-x_j L_j)\exp((x_k + \delta x_k)L_k). \tag{A23}$$

The CH formula gives

$$\delta a = \delta x_j L_j - \sum_{j,k,i=1}^{3} \delta x_j x_i \epsilon_{jik} L_k - \sum_{j,k=1}^{3} \delta x_j \frac{(x_j x_k - x^2\delta_{jk})}{x^2}\left(\frac{x}{2}\cot\frac{x}{2} - 1\right)L_k \tag{A24}$$

where $x^2 = \Sigma_{j=1}^3 x_j^2$. Hence

$$e_{jk} = \delta_{jk} - \sum_{i=1}^{3} \tfrac{1}{2} x_i \epsilon_{jik} - \left(\frac{x}{2}\cot\frac{x}{2} - 1\right)\left(\frac{x_j x_k - x^2\delta_{jk}}{x^2}\right) \qquad \text{for } 1 \leqslant j,\ k \leqslant 3 \tag{A25}$$

and

$$e_{4j} = \delta_{4j} \tag{A26}$$

$$e_{5j} = \delta_{5j}. \tag{A27}$$

References

Anderson P W 1988 *Frontiers and Borderlines in Many Particle Physics* ed R A Broglia and J R Schrieffer (Amsterdam: North-Holland)

Bars I 1985 *Physica* **15D** 42

Bars I and Günaydin M 1983 *Commun. Math. Phys.* **91** 31

Berezin F A 1987 *Introduction to Superanalysis* ed A A Kirillov (Dordrecht: Reidel)

Bickers N E 1987 *Rev. Mod. Phys.* **59** 845

Cornwell J F 1989 *Supersymmetries and Infinite Dimensional Algebras* (*Group Theory in Physics 3*) (New York: Academic)

Fradkin E and Stone M 1988 *Phys. Rev.* B **38** 7215

Georgi H 1982 *Lie Algebras in Particle Physics* (New York: Benjamin)

Hubbard J 1965 *Proc. R. Soc.* A **285** 542

Kac V G 1977 *Commun. Math. Phys.* **53** 31

Klauder J R and Skagerstam B-S 1985 *Coherent States* (Singapore: World Scientific)

Montorsi A, Rasetti M and Solomon Al 1989a *Int. J. Mod. Phys.* B **2** 247

—— 1989b *Proc. 17th Int. Colloq. on Group Theoretical Methods in Physics* ed Y Saint-Aubin and L Vinet (Singapore: World Scientific)

—— 1989c *Proc. NATO Conf. on Interacting Electrons in Reduced Dimensions* ed D Baeriswyl and D K Campbell (New York: Plenum)

Mott N F 1974 *Metal–Insulator Transitions* (London: Taylor and Francis)

Perelomov A 1986 *Generalised Coherent States and their Applications* (Berlin: Springer)

Pike E R, Jefferson J H and Sarkar S 1991 *An Introduction to Electron Correlations and High Temperature Superconductivity* (Bristol: Hilger) to be published

Sarkar S 1990 *J. Phys. A: Math. Gen.* **23** L409

—— 1991 *J. Phys. A: Math. Gen.* **24** 1137

Sourlas N 1985 *Physica* **15D** 115

Wess J and Bagger J 1983 *Supersymmetry and Supergravity* (Princeton)

West P C 1986 *Introduction to Supersymmetry and Supergravity* (Singapore: World Scientific)

Wiegmann P B 1988 *Phys. Rev. Lett.* **60** 821